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Continuous Optimization

A nonlinear interval number programming method for uncertain optimization problems

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Abstract

In this paper, a method is suggested to solve the nonlinear interval number programming problem with uncertain coefficients both in nonlinear objective function and nonlinear constraints. Based on an order relation of interval number, the uncertain objective function is transformed into two deterministic objective functions, in which the robustness of design is considered. Through a modified possibility degree, the uncertain inequality and equality constraints are changed to deterministic inequality constraints. The two objective functions are converted into a single-objective problem through the linear combination method, and the deterministic inequality constraints are treated with the penalty function method. The intergeneration projection genetic algorithm is employed to solve the finally obtained deterministic and non-constraint optimization problem. Two numerical examples are investigated to demonstrate the effectiveness of the present method.

Keywords: Uncertain optimization; Nonlinear programming; Interval number; Genetic algorithm

1. Introduction

In traditional mathematical programming, the coefficients of the problems are always treated as deterministic values. However uncertainty always exits in practical engineering problems. In order to deal with the uncertain optimization problems, fuzzy and stochastic approaches are commonly used to describe the imprecise characteristics. In stochastic programming (e.g. Charnes and Cooper, 1959; Kall, 1982; Liu et al., 2003; Cho, 2005) the uncertain coefficients are regarded as random variables and their probability distributions are assumed to be known. In fuzzy programming (e.g. Slowinski, 1986; Delgado et al., 1989; Luhandjula, 1989; Liu and Iwamura, 2001) the constraints and objective function are viewed as fuzzy sets and their membership functions also need to be known. In these two kinds of approaches, the membership functions and probability

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distributions play important roles. However, it is sometimes difficult to specify an appropriate membership function or accurate probability distribution in an uncertain environment (Sengupta et al., 2001).

In recent years, the interval analysis method was developed to model the uncertainty in uncertain optimization problems, in which the bounds of the uncertain coefficients are only required, not necessarily knowing the probability distributions or membership functions. Tanaka et al. (1984) and Rommelfanger (1989) discussed the linear programming problem with interval coefficients in objective function. Chanas and Kuchta (1996a,b) suggested an approach based on an order relation of interval number to convert the linear optimization problem with uncertainty into a deterministic optimization problem. Tong (1994) investigated the problems in which the coefficients of the objective function and the constraints are all interval numbers. He obtained the possible interval of the solution by taking the maximum value range and minimum value range inequalities as constraint conditions. Liu and Da (1999) proposed an interval number optimization method based on a fuzzy constraint satisfactory degree to deal with the linear problems. Sengupta et al. (2001) studied the linear interval number programming problems in which the coefficients of the objective function and inequality constraints are all interval numbers. They proposed the concept of "acceptability index" and gave one solution for the uncertain linear programming. Zhang et al. (1999) assumed interval numbers as random variables with uniform distributions and constructed a possibility degree to solve the multi-criteria decision problem. The above methods point out a fine way for the uncertain optimization. However, only linear programming problems are investigated. For most of the engineering problems, the objective function and constraints are nonlinear, and they are always obtained through numerical algorithms such as finite element method (FEM) instead of the explicit expressions. Furthermore, only the linear inequality constraints are studied and they have not proposed an approach to deal with the uncertain equality constraints. The reference (Ma, 2002), on the best knowledge of the authors, seems the first and only publication on nonlinear interval number programming (NINP). In this reference, a deterministic optimization method is used to obtain the interval of the nonlinear objective function, and a three-objective optimization problem is formulated. However, only the uncertain objective function is considered, and no approach is proposed to deal with the nonlinear constraints with uncertainty. As a result, an effective method still have not been developed to deal with the general NINP problem in which there exit not only uncertain nonlinear objective function but also uncertain nonlinear constraints, so far.

In this paper, an NINP method is firstly suggested to deal with the general nonlinear optimization problems. An order relation of interval number is used to transform the uncertain single-objective optimization into a deterministic two-objective optimization, which considers the midpoint and radius of the uncertain objective function simultaneously. A modified possibility degree of interval number based on the probability method is suggested to deal with the uncertain inequality constraints. The uncertain equality constraint is firstly investigated, and it is solved by being transformed into two uncertain inequality constraints. For each specific decision vector, two deterministic optimization processes are performed to obtain the interval of the objective function or constraint. An unconstraint and single-objective optimization problem is finally formulated through the linear combination method and the penalty function method. The intergeneration projection genetic algorithm (IP-GA) with fine global convergence performance is employed as optimization solver. A benchmark test is presented to demonstrate the effectiveness of the present method, and then this method is applied to a practical engineering problem, namely the locators' optimization of an automobile welding fixture with uncertain load and material property.

2. Statement of the problem

A general nonlinear interval number programming (NINP) problem with uncertain interval coefficients in both of the objective function and constraints can be given as follows:

$$\begin{aligned} \min \quad & f(\mathbf{X},\mathbf{U}) \\ \text{s.t.} \quad & g_i(\mathbf{X},\mathbf{U}) \geqslant (=,\leqslant)[v_i^{\mathrm{L}}, v_i^{\mathrm{R}}], \quad i=1,\ldots,l, \\ & \mathbf{X}\in\Omega^n, \quad \mathbf{U}=[\mathbf{U}^{\mathrm{L}},\mathbf{U}^{\mathrm{R}}], \quad U_i=[U_i^{\mathrm{L}},U_i^{\mathrm{R}}], \ \ i=1,2,\ldots,q, \end{aligned} \tag{1}$$

where $\mathbf{X}$ is an n-dimensional decision vector and Ω^n is its range. f and g_i are objective function and the ith constraint, respectively, and l is the number of the constraints. They are all nonlinear functions of $\mathbf{X}$ and $\mathbf{U}$, and continuous at $\mathbf{U}$. $\mathbf{U}$ is a q-dimensional uncertain vector and its components are all interval numbers. $[U_i^{\mathrm{L}}, U_i^{\mathrm{R}}]$ denotes an interval number and it represents a bounded set of real numbers between the bounds. The superscripts L and R denote lower and upper bounds of an interval number, respectively. $[v_i^{\mathrm{L}}, v_i^{\mathrm{R}}]$ denotes the allowable interval number of the ith constraint. For each specific $\mathbf{X}$, the possible values of the objective function or a constraint will form an interval instead of a real number. Furthermore the objective function and constraints are all nonlinear, thus the traditional deterministic optimization methods and the linear interval number programming methods cannot be used to solve this problem. In the following sections, a new method will be suggested to solve the above problem.

3. The optimization procedure

3.1. Treatment of the uncertain objective function

In interval mathematics, an order relation is often used to rank interval numbers, and it implies that an interval number is better than another but not that one is larger than another. Ishibuchi and Tanaka (1990) defined an order relation $\leqslant_{mw}$ between interval numbers A and B for maximization problems as follows:

$$\begin{aligned}
&A \leqslant_{mw} B \quad \text{if } m(A) \leqslant m(B) \text{ and } w(A) \geqslant w(B),\\
&A <_{mw} B \quad \text{if } A \leqslant_{mw} B \text{ and } A \neq B,\\
&m(A) = \frac{A^{\mathrm{L}} + A^{\mathrm{R}}}{2}, \quad w(A) = \frac{A^{\mathrm{R}} - A^{\mathrm{L}}}{2}, \quad m(B) = \frac{B^{\mathrm{L}} + B^{\mathrm{R}}}{2}, \quad w(A) = \frac{B^{\mathrm{R}} - B^{\mathrm{L}}}{2},
\end{aligned} \tag{2}$$

where $\leqslant_{mw}$ represents a preference of the decision maker to the midpoint value m and the radius w of the interval number. For minimization problems, Eq. (2) has the following form:

$$\begin{aligned}
&A \leqslant_{mw} B \quad \text{if } m(A) \geqslant m(B) \text{ and } w(A) \geqslant w(B),\\
&A <_{mw} B \quad \text{if } A \leqslant_{mw} B \text{ and } A \neq B.
\end{aligned} \tag{3}$$

Using $\leqslant_{mw}$ to compare the objective function in Eq. (1), we hope that the interval of the objective function caused by the uncertainty has not only a small midpoint but also a small radius. Thus the uncertain objective function can be transformed into a two-objective optimization problem as follows:

$$\begin{aligned}
&\min[m(f(\mathbf{X},\mathbf{U})), w(f(\mathbf{X},\mathbf{U}))],\\
&m(f(\mathbf{X},\mathbf{U})) = \tfrac{1}{2}(f^{\mathrm{L}}(\mathbf{X}) + f^{\mathrm{R}}(\mathbf{X})),\\
&w(f(\mathbf{X},\mathbf{U})) = \tfrac{1}{2}(f^{\mathrm{R}}(\mathbf{X}) - f^{\mathrm{L}}(\mathbf{X})).
\end{aligned} \tag{4}$$

To a certain $\mathbf{X}$, $f(\mathbf{X},\mathbf{U})$ is an interval number, and its bounds $f^{\mathrm{L}}(\mathbf{X})$, $f^{\mathrm{R}}(\mathbf{X})$ can be obtained (Ma, 2002):

$$\begin{aligned}
&f^{\mathrm{L}}(\mathbf{X}) = \min_{\mathbf{U}\in\Gamma} f(\mathbf{X},\mathbf{U}), \qquad f^{\mathrm{R}}(\mathbf{X}) = \max_{\mathbf{U}\in\Gamma} f(\mathbf{X},\mathbf{U}),\\
&\mathbf{U} \in \Gamma = \{\mathbf{U} | \mathbf{U}^{\mathrm{L}} \leqslant \mathbf{U} \leqslant \mathbf{U}^{\mathrm{R}}\}.
\end{aligned} \tag{5}$$

Through Eq. (5), the uncertain vector $\mathbf{U}$ is eliminated, and hence the two objective functions in Eq. (4) become deterministic.

The two objective functions in Eq. (4) are analogous to minimize the average value and the deviation of the uncertain objective function, respectively. Through minimizing the deviation, the variance of the objective function caused by the uncertainty will be decreased, namely, the optimal design can make the uncertain objective function insensitive to the fluctuation of the uncertain coefficients. Thus a robustness of the design can be guaranteed.

3.2. Treatment of the uncertain inequality constraints

The possibility degree of interval number represents certain degree that one interval number is larger or smaller than another. Reference (Zhang et al., 1999) extended probability method into the comparison of interval numbers, and proposed a construction method for the possibility degree $P_{A\geqslant B}$ and $P_{B\geqslant A}$ based on three relations of interval numbers A and B as shown in Fig. 1:

$$P_{A\geqslant B}=\begin{cases}1, & A^{\mathrm{L}}\geqslant B^{\mathrm{R}},\\ \frac{A^{\mathrm{R}}-B^{\mathrm{R}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}+\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}\cdot\frac{A^{\mathrm{L}}-B^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}+0.5\cdot\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}\cdot\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}, & B^{\mathrm{L}}\leqslant A^{\mathrm{L}}<B^{\mathrm{R}}\leqslant A^{\mathrm{R}},\\ \frac{A^{\mathrm{R}}-B^{\mathrm{R}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}+0.5\cdot\frac{B^{\mathrm{R}}-B^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}, & A^{\mathrm{L}}<B^{\mathrm{L}}<B^{\mathrm{R}}\leqslant A^{\mathrm{R}},\end{cases} \tag{6}$$

$$P_{B\geqslant A}=\begin{cases}0, & A^{\mathrm{L}}\geqslant B^{\mathrm{R}},\\ 0.5\cdot\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}\cdot\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}, & B^{\mathrm{L}}\leqslant A^{\mathrm{L}}<B^{\mathrm{R}}\leqslant A^{\mathrm{R}},\\ \frac{B^{\mathrm{L}}-A^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}+0.5\cdot\frac{B^{\mathrm{R}}-B^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}, & A^{\mathrm{L}}<B^{\mathrm{L}}<B^{\mathrm{R}}\leqslant A^{\mathrm{R}}.\end{cases} \tag{7}$$

Here interval numbers A and B are regarded as random variables $\widetilde{A}$ and $\widetilde{B}$ with uniform distributions in their intervals. The probability for random variable $\widetilde{A}$ larger or smaller than $\widetilde{B}$ is regarded as $P_{A\geqslant B}$ or $P_{B\geqslant A}$. Comparing with the possibility degree based on a fuzzy set (e.g. Sengupta et al., 2001), this method can give the possibility degree of interval number a more intuitive and mathematical description. However, using Eqs. (6) and (7) to calculate the possibility degree of the same two interval numbers will make the following treatment of the inequality constraints inconvenient. Fig. 2 lists all of the possible relations between A and B and based on these relations a modified possibility degree is suggested using above probability method:

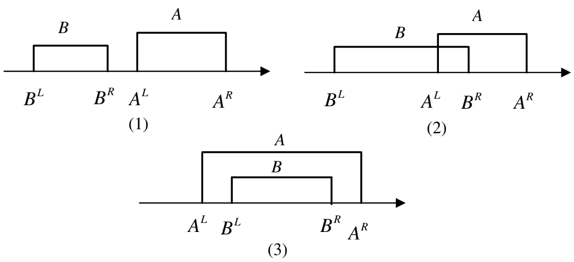

Fig. 1. Three relations between intervals A and B.

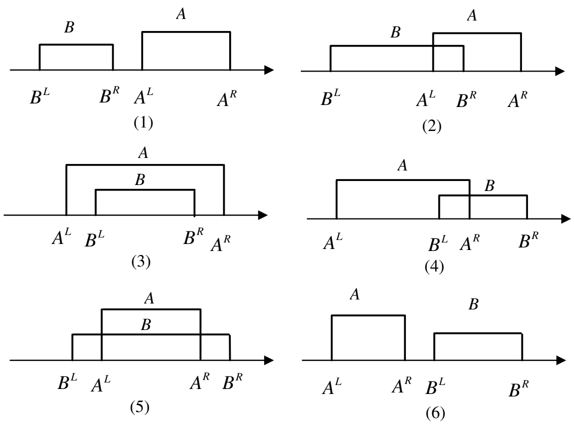

Fig. 2. Six possible relations between intervals A and B.

$$P_{A\geqslant B}=\begin{cases}1, & A^{\mathrm{L}}\geqslant B^{\mathrm{R}},\\ \frac{A^{\mathrm{R}}-B^{\mathrm{R}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}+\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}\cdot\frac{A^{\mathrm{L}}-B^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}+0.5\cdot\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}\cdot\frac{B^{\mathrm{R}}-A^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}, & B^{\mathrm{L}}\leqslant A^{\mathrm{L}}<B^{\mathrm{R}}\leqslant A^{\mathrm{R}},\\ \frac{A^{\mathrm{R}}-B^{\mathrm{R}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}+0.5\cdot\frac{B^{\mathrm{R}}-B^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}, & A^{\mathrm{L}}<B^{\mathrm{L}}<B^{\mathrm{R}}\leqslant A^{\mathrm{R}},\\ 0.5\cdot\frac{A^{\mathrm{R}}-B^{\mathrm{L}}}{A^{\mathrm{R}}-A^{\mathrm{L}}}\cdot\frac{A^{\mathrm{R}}-B^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}, & A^{\mathrm{L}}<B^{\mathrm{L}}\leqslant A^{\mathrm{R}}<B^{\mathrm{R}},\\ \frac{A^{\mathrm{L}}-B^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}+0.5\cdot\frac{A^{\mathrm{R}}-A^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}, & B^{\mathrm{L}}\leqslant A^{\mathrm{L}}<A^{\mathrm{R}}<B^{\mathrm{R}},\\ 0, & A^{\mathrm{R}}<B^{\mathrm{L}}.\end{cases}\tag{8}$$

When B is degenerated into a real number b, all of the possible relations between A and b is shown in Fig. 3. Here only A is assumed as a random variable $\widetilde{A}$ with uniform distribution, and the degenerated possibility degree $P_{A\geqslant b}$ can be written:

$$P_{A\geqslant b}=\begin{cases}0, & b>A^{\mathrm{R}},\\ \frac{A^{\mathrm{R}}-b}{A^{\mathrm{R}}-A^{\mathrm{L}}}, & A^{\mathrm{L}}<b\leqslant A^{\mathrm{R}},\\ 1, & b\leqslant A^{\mathrm{L}}.\end{cases}\tag{9}$$

In the same way, the possibility degree $P_{a\geqslant B}$ can be also obtained based on Fig. 4 when A is degenerated into a real number a:

$$P_{a\geqslant B}=\begin{cases}0, & a<B^{\mathrm{L}},\\ \frac{a-B^{\mathrm{L}}}{B^{\mathrm{R}}-B^{\mathrm{L}}}, & B^{\mathrm{L}}\leqslant a<B^{\mathrm{R}},\\ 1, & a\geqslant B^{\mathrm{R}}.\end{cases}\tag{10}$$

From the above equations, it can be found that $P_{A\geqslant B}=\alpha$ $(0\leqslant\alpha\leqslant 1)$ means that A is larger than B with the degree α. $\alpha=1$ implies that A is always larger than B and $\alpha=0$ implies that A is always smaller than B.

In stochastic optimization (e.g. Liu et al., 2003), we often make the constraints satisfied with a confidence level and transform the uncertain constraints into deterministic constraints. Similarly, we can make the inequality constraint $g_i(\mathbf{X},\mathbf{U})\geqslant[v_i^{\mathrm{L}},v_i^{\mathrm{R}}]$ in Eq. (1) satisfied with a possibility degree level, and formulate a deterministic inequality:

$$P_{C\geqslant D}\geqslant\lambda_i,\quad C=[g_i^{\mathrm{L}}(\mathbf{X}),g_i^{\mathrm{R}}(\mathbf{X})],\ \ D=[v_i^{\mathrm{L}},v_i^{\mathrm{R}}],\tag{11}$$

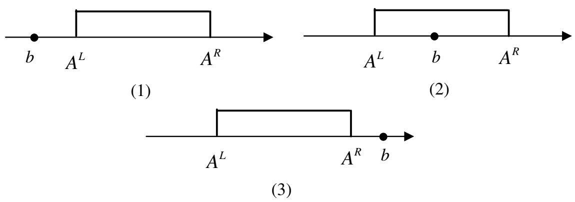

Fig. 3. Three possible relations between real number b and interval A.

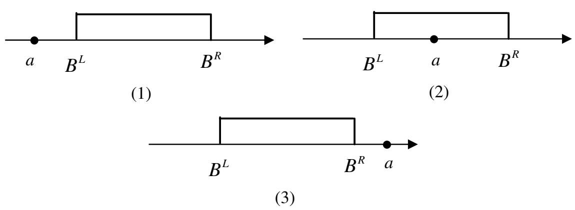

Fig. 4. Three possible relations between real number a and interval B.

where $P_{C\geqslant D}$ is the possibility degree of the ith constraint. $0 \leqslant \lambda_i \leqslant 1$ is a predetermined possibility degree level. C is the interval of the constraint function at $\mathbf{X}$ and its bounds can be obtained through two deterministic optimization processes:

$$g_i^{\mathrm{L}}(\mathbf{X}) = \min_{U\in\Gamma} g_i(\mathbf{X},\mathbf{U}), \qquad g_i^{\mathrm{R}}(\mathbf{X}) = \max_{U\in\Gamma} g_i(\mathbf{X},\mathbf{U}), \tag{12}$$

λ_i can be adjusted to control the feasible field of $\mathbf{X}$. When λ_i is larger, the inequality constraint equation (11) will be restricted more strictly and the feasible field of $\mathbf{X}$ will become smaller. For the inequality constraint $g_i(\mathbf{X},\mathbf{U}) \leqslant [v_i^{\mathrm{L}}, v_i^{\mathrm{R}}]$, it can be changed to $[v_i^{\mathrm{L}}, v_i^{\mathrm{R}}] \geqslant g_i(\mathbf{X},\mathbf{U})$ and treated with the above method.

3.3. Treatment of the uncertain equality constraints

For the equality constraint $g_i(\mathbf{X},\mathbf{U}) = [v_i^{\mathrm{L}}, v_i^{\mathrm{R}}]$, it can be transformed into the following form:

$$v_i^{\mathrm{L}} \leqslant g_i(\mathbf{X},\mathbf{U}) \leqslant v_i^{\mathrm{R}}. \tag{13}$$

Then Eq. (13) can be converted into the following two uncertain inequality constraints:

$$\begin{cases} g_i(\mathbf{X},\mathbf{U}) \geqslant v_i^{\mathrm{L}}, \\ v_i^{\mathrm{R}} \geqslant g_i(\mathbf{X},\mathbf{U}). \end{cases} \tag{14}$$

Predetermining two possibility degree levels λ_{i1} and λ_{i2}, Eq. (14) can be changed as follows:

$$\begin{gathered} \begin{cases} P_{G\geqslant e} \geqslant \lambda_{i1}, \\ P_{h\geqslant G} \geqslant \lambda_{i2}, \end{cases} \\ G = [g_i^{\mathrm{L}}(\mathbf{X}), g_i^{\mathrm{R}}(\mathbf{X})] = \left[\min_{U\in\Gamma} g_i(\mathbf{X},\mathbf{U}), \max_{U\in\Gamma} g_i(\mathbf{X},\mathbf{U})\right], \;\; e = v_i^{\mathrm{L}}, \;\; h = v_i^{\mathrm{R}}, \end{gathered} \tag{15}$$

where G is the interval of the equality constraint. The possibility degrees $P_{G\geqslant e}$ and $P_{h\geqslant G}$ can be calculated through Eqs. (9) and (10). Thus, the uncertain equality constraint has been converted into two deterministic inequality constraints.

3.4. Deterministic optimization

Through the above treatments, the NINP problem equation (1) is transformed into the following deterministic two-objective programming problem:

$$\begin{aligned} \min \quad & [m(f(\mathbf{X},\mathbf{U})), w(f(\mathbf{X},\mathbf{U}))] \\ \text{s.t.} \quad & P_{M_i\geqslant N_i} \geqslant \lambda_i, \quad i = 1,2,\ldots,k, \\ & m(f(\mathbf{X},\mathbf{U})) = \tfrac{1}{2}\left(\min_{\mathbf{U}\in\Gamma} f(\mathbf{X},\mathbf{U}) + \max_{\mathbf{U}\in\Gamma} f(\mathbf{X},\mathbf{U})\right), \\ & w(f(\mathbf{X},\mathbf{U})) = \tfrac{1}{2}\left(\max_{\mathbf{U}\in\Gamma} f(\mathbf{X},\mathbf{U}) - \min_{\mathbf{U}\in\Gamma} f(\mathbf{X},\mathbf{U})\right), \\ & \mathbf{X} \in \Omega^n, \;\; \mathbf{U} \in \Gamma = \{\mathbf{U} | \mathbf{U}^{\mathrm{L}} \leqslant \mathbf{U} \leqslant \mathbf{U}^{\mathrm{R}}\}, \;\; \lambda_i \in [0,1], \end{aligned} \tag{16}$$

where M_i and N_i can be both intervals or one of them is a real number. k is the number of the uncertain inequality constraints after transformation of the uncertain equality constraints.

The linear combination method (Hu, 1990) is adopted to deal with the multi-objective optimization. In multi-objective optimization, applying the linear combination method to integrate the objective functions is a relatively easy and convenient way, provided that the preferences of the objective functions are available. In addition, the linear combination method is easy to be realized in computer program. On the other hand, the penalty function method (Chen, 2002) is used to deal with the constraints. Thus Eq. (16) can be transformed into a non-constraint and single-objective optimization problem in terms of a penalty function $\tilde{f}$:

$$\min \tilde{f} = (1-\beta)(m(f(\mathbf{X},\mathbf{U}))+\xi)/\phi + \beta(w(f(\mathbf{X},\mathbf{U}))+\xi)/\psi + \sigma\sum_{i=1}^{k}\varphi(P_{M_i \geqslant N_i}-\lambda_i),$$
$$\phi = \min_{\mathbf{X}\in\Omega^n}(m(f(\mathbf{X},\mathbf{U}))+\xi),\quad \psi = \min_{\mathbf{X}\in\Omega^n}(w(f(\mathbf{X},\mathbf{U}))+\xi), \tag{17}$$

where $0 \leqslant \beta \leqslant 1$ is a weighting factor of the two objective functions. ξ is a number which makes $m(f(\mathbf{X},\mathbf{U}))+\xi$ and $w(f(\mathbf{X},\mathbf{U}))+\xi$ non-negative. ϕ and ψ are two normalization factors. σ is the penalty factor which is usually specified as a large value. φ is a function which has the following form:

$$\varphi(P_{M_i \geqslant N_i}-\lambda_i) = (\max(0, -(P_{M_i \geqslant N_i}-\lambda_i)))^2. \tag{18}$$

For each optimization step of solving the nonlinear optimization equation (17), the sub-optimization processes are involved. The traditional gradient-based optimization algorithms probably come into the trap of local optimum. In addition, for most of the practical engineering problems the optimization model cannot be expressed explicitly, and hence the information on derivatives is difficult to obtain accurately. Considering that GA has a fine global convergence performance and only needs the information of functional values, it seems an appropriate selection for the present problem.

4. IP-GA and computational procedure

IP-GA (Xu et al., 2001) is a modification based on the micro GA (μGA) (Krishnakumar, 1989). The μGA is an extension of conventional GAs, and it is capable of avoiding the premature convergence and performing better in reaching the optimal region than traditional GAs. In general, the population size of μGA is very small around 5–8. This small population size can guarantee a fast convergence to a local optimum. For maintaining the genetic diversity, a restart strategy is used instead of the conventional mutation operation. That is, once the current generation converges, a new generation will be generated which has the same population size and consists of the best individual from the previously converged generation and other newly generated individuals based on a random number function. The intergeneration projection (IP) operator aims to find a better individual by jumping along the move direction of the best individuals at two consecutive generations so as to improve the convergence rate. The IP operator produces three new child individuals c_1, c_2 and c_3 based on the following equation (Xu et al., 2001):

$$c_1 = p_j^b + r(p_j^b - p_{j-1}^b),\quad c_2 = p_{j-1}^b + s(p_j^b - p_{j-1}^b),\quad c_3 = p_j^b - t(p_j^b - p_{j-1}^b), \tag{19}$$

where p_j^b, p_{j-1}^b are the best individuals of the current (parent) and the previous (grandparent) generations, respectively. $0 \leqslant r \leqslant 1$, $0 \leqslant s \leqslant 1$ and $0 \leqslant t \leqslant 1$ are three non-negative parameters whose values can be changed to adjust the distances of these new individuals to original individuals p_j^b and p_{j-1}^b. Then the three newly obtained individuals are used to replace the three worst individuals in the present offspring. IP-GA combines the μGA with IP operator and whereby has a better global convergence performance. In the literature (Liu and Han, 2003), the convergence performance of IP-GA has been fully investigated based on six benchmark tests including two complex multi-modal functions. It is found that IP-GA has a much better global convergence performance than μGA. To arbitrary combinations of r, s and t, the generation number that reaches the global optimum are only 6.4–74.4% of μGA.

The computational flowchart is shown in Fig. 5. The presented computational procedure involves the nesting of IP-GA. The outer IP-GA is used to optimize the penalty function $\tilde{f}$ and the inner IP-GA is used to obtain the intervals of the objective function and constraints. In the outer IP-GA, each individual chromosome represents a candidate decision vector $\mathbf{X}$. For each $\mathbf{X}$, the intervals of the objective function and the constraints can be calculated through the inner IP-GA in which $\mathbf{X}$ is specified as a constant and $\mathbf{U}$ as the decision vector. For computing each interval of the objective function or a constraint, the outer IP-GA will call for two cycles of the inner IP-GA. Then the midpoint value and radius of the objective function, and the possibility

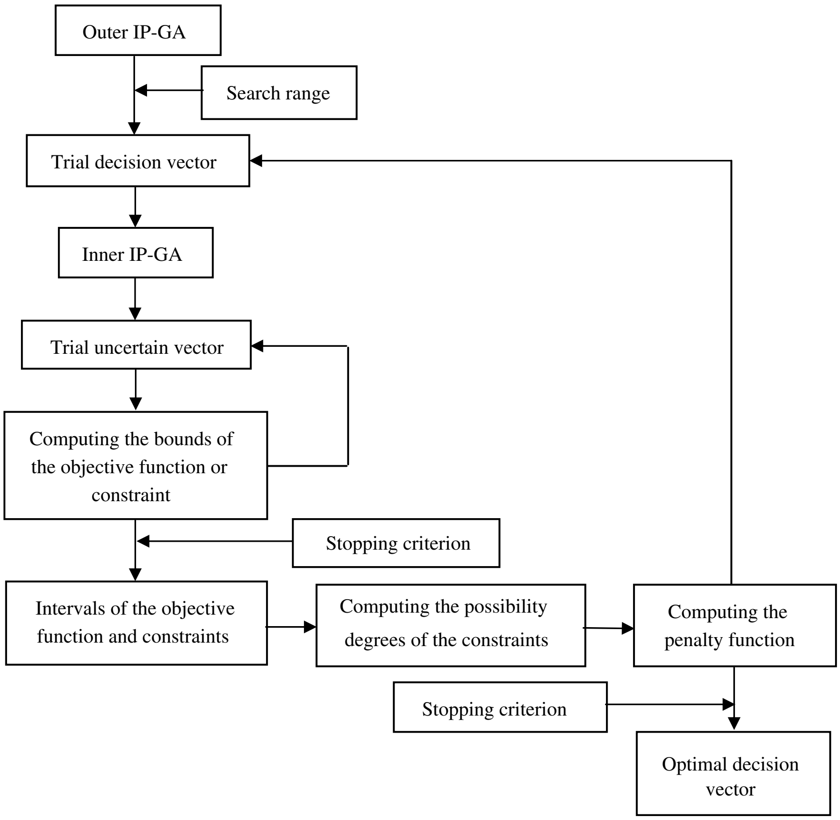

Fig. 5. Computational procedure of NINP based on IP-GA.

degrees of the constraints are calculated. The maximum generations are employed as stopping criterion for IP-GA.

5. Numerical examples and discussion

5.1. Benchmark test

A benchmark function is provided to test the performance of the present method:

$$\begin{aligned}
\min \quad & f[\mathbf{X},\mathbf{U}] = U_1(X_1-2)^2 + U_2(X_2-1)^2 + U_3X_3 \\
\text{s.t.} \quad & U_1X_1^2 - U_2^2X_2 + U_3X_3 = [6.5, 7.0], \\
& U_1X_1 + U_2X_2 + U_3^2X_3^2 + 1 \geqslant [10.0, 15.0], \\
& -1 \leqslant X_1 \leqslant 5, \quad -3 \leqslant X_2 \leqslant 6, \quad -2 \leqslant X_3 \leqslant 7, \\
& U_1 = [1.0, 1.3], \quad U_2 = [0.9, 1.1], \quad U_3 = [1.2, 1.4].
\end{aligned} \tag{20}$$

The equality constraint is separated into two inequality constraints according to Eq. (15) and their possibility degrees are denoted by P_1 and P_2, respectively. The possibility degree of the inequality constraint is denoted by P_3. ξ is specified as 4.0. The normalization factors ϕ and ψ are specified as 1.4 and 2.0 based on Eq. (17), respectively. The weighting factor β is set to 0.5 which means that the two objective functions are given a same preference. The penalty factor σ is set to 1000. For IP-GA, The search parameters r, s, t, the population size and the probability of crossover are set to 0.6, 0.6, 0.6, 5 and 0.5, respectively.

To analyze the convergence performance of the present method, different maximum generations with 100, 200, 300, 400, 500 and 600 are investigated, respectively. Because GA uses random numbers to identify trial individuals, result of a single GA run can be a matter of chance. It can be expected that the statistical mean of some GA runs can provide higher confidence about the global solution obtained. In IP-GA, the random number function generates different random numbers under the different random number seeds (Liu and Han, 2003). Therefore, for each generation number four GA runs with different random number seeds −1000, −5000, −10,000 and −15,000 are performed, and the mean values of these results are obtained as the optimal solutions as shown in Tables 1–4. It is found that the optimization results reach the stationary values when the generation number is 400, and they keep same with increasing of the generation number from 400 to 600. It is because that for this problem generation number 400 is enough to obtain the stable global optimums. Comparing with the results of generation number 400, the results of generation number 100 is relatively coarse, and it implies that 100 generations are not enough to reach the fine optimums. For generation number 200, the

Table 1
Optimization results under different possibility degree levels (100 generations)

λ	P_1	P_2	P_3	Penalty function $\tilde{f}$	The optimum (X_1, X_2, X_3)
0.0	0.00	1.00	0.21	1.70	(2.10, 0.88, −1.98)
0.2	0.43	0.79	0.31	2.12	(2.90, 0.57, −1.98)
0.4	0.50	0.79	0.91	3.95	(1.93, 0.99, 2.48)
0.6	0.58	0.57	0.74	4.50	(2.30, 2.05, 2.22)
0.8	0.73	0.72	1.00	15.62	(0.16, −0.14, 5.20)
1.0	0.71	0.70	1.00	171.23	(−0.33, 0.22, 5.31)

Table 2
Optimization results under different possibility degree levels (200 generations)

λ	P_1	P_2	P_3	Penalty function $\tilde{f}$	The optimum (X_1, X_2, X_3)
0.0	0.00	1.00	0.25	1.62	(2.09, 0.96, −1.98)
0.2	0.35	0.87	0.33	1.99	(2.85, 0.96, −1.97)
0.4	0.67	0.60	0.90	3.82	(2.05, 1.03, 2.48)
0.6	0.58	0.59	0.70	4.12	(2.31, 1.87, 2.21)
0.8	0.73	0.72	1.00	15.15	(0.16, 0.03, 5.32)
1.0	0.70	0.72	1.00	160.08	(−0.26, 0.07, 5.15)

Table 3
Optimization results under different possibility degree levels (300 generations)

λ	P_1	P_2	P_3	Penalty function $\tilde{f}$	The optimum (X_1, X_2, X_3)
0.0	0.00	1.00	0.22	1.57	(2.05, 0.92, −2.00)
0.2	0.34	0.83	0.31	1.92	(2.84, 0.84, −1.93)
0.4	0.59	0.65	0.90	3.72	(1.96, 1.01, 2.49)
0.6	0.60	0.59	0.69	3.95	(2.26, 1.88, 2.17)
0.8	0.74	0.75	1.00	14.81	(0.20, −0.02, 5.17)
1.0	0.75	0.72	1.00	156.00	(−0.28, 0.09, 5.21)

Table 4
Optimization results under different possibility degree levels (400, 500 and 600 generations)

λ	P_1	P_2	P_3	Penalty function $\tilde{f}$	The optimum (X_1, X_2, X_3)
0.0	0.00	1.00	0.22	1.57	(2.05, 0.92, −2.00)
0.2	0.34	0.83	0.31	1.92	(2.84, 0.84, −1.93)
0.4	0.59	0.65	0.90	3.72	(1.96, 1.01, 2.49)
0.6	0.60	0.59	0.69	3.95	(2.26, 1.88, 2.17)
0.8	0.74	0.75	1.00	14.79	(0.20, −0.02, 5.17)
1.0	0.75	0.72	1.00	159.90	(−0.28, 0.09, 5.21)

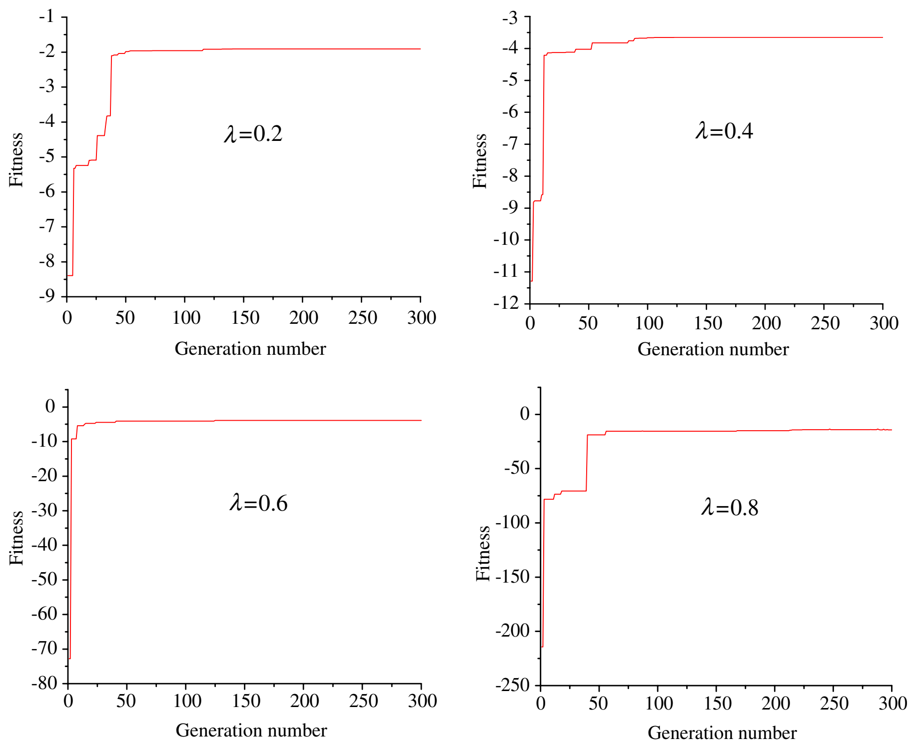

Fig. 6. Convergence performance of IP-GA in the NINP optimization.

optimization results become better. For generation number 300, the optimization results are nearly equal to the ones of generation number 400. From Table 4, it is found that the optimums are different at different λ. When $\lambda = 1.0$, the penalty function $\tilde{f}$ has the largest value 159.90, and the optimal X_i, $i = 1, 2, 3$ are -0.28, 0.09 and 5.21, respectively. With the variation of λ from 1.0 to 0.0, $\tilde{f}$ becomes smaller and smaller. When $\lambda = 0.0$, it reaches a best value 1.57, and the optimal X_i, $i = 1, 2, 3$ are 2.05, 0.92 and -2.00, respectively. This is because that a larger λ makes the constraints stricter and it will lead to a smaller feasible field of the decision vector and whereby a worse $\tilde{f}$. The convergence performance of IP-GA at $\lambda = 0.2$, 0.4, 0.6 and 0.8 under random number seed $-10{,}000$ for generation number 300 is shown in Fig. 6. The fitness is the negative value of the penalty function. It can be found that the iterative solutions of IP-GA can approach the optimum very quickly and hence has a high convergence velocity.

The optimization results also indicate that the design objective and the possibility degrees of the constraints are always contradictive. A better design objective is at the price of smaller possibility degrees of the constraints. However, a smaller possibility degree means a larger possibility that the constraint will be violated. Thus in NINP an appropriate trade-off between these two factors should be made through adjusting the possibility degree level of the constraints.

5.2. Application

A practical engineering problem to optimize the locators of an automobile welding fixture as shown in Fig. 7 is considered. The workpiece is an L-shaped sheet metal. q_1 and q_2 are two locating holes with $\varnothing 15$ mm and their positions are (80 mm, 80 mm, 0) and (420 mm, 320 mm, 0), respectively. L_1, L_2 and L_3 are three locators. F is a spot-welding force caused by the welding gun and E is its loading point whose position is (500 mm, 50 mm, 25 mm). In the assembling process, the sheet metal should be located precisely through the welding fixture and then connected by spot welding. Because the sheet metal is easy to deform, the design of the welding fixture is different from the traditional fixtures. The welding fixture should be able

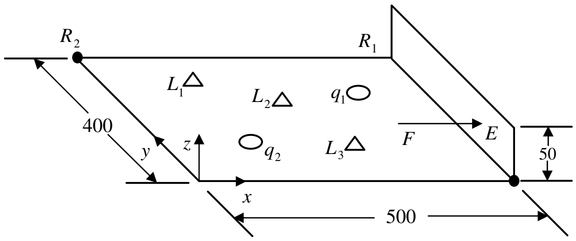

Fig. 7. Locating model of an automobile welding fixture (mm).

to not only locate the part precisely but also restrict the excessive elastic deformation of the part. The spot-welding force F, the Young's modulus and Poisson's ration of the sheet metal are uncertain, and their intervals are [48 N, 53 N], [1.9×10^{11} Pa, 2.2×10^{11} Pa] and [0.3, 0.4], respectively. In this numerical example, the positions of the locators need to be optimized to obtain a minimum deformation of the sheet metal. Five key points located in (250 mm, 0, 0), (500 mm, 200 mm, 0), (250 mm, 400 mm, 0), (0, 200 mm, 0) and (250 mm, 200 mm, 0) are selected, and the sum of their displacements in z direction is used to represent the deformation of the entire sheet metal. In addition, two constraints are added to the points R_1 and R_2 that their displacements in z direction should be less than [1 mm, 1.3 mm] and [0.9 mm, 1.2 mm], respectively. As a result, the optimization model can formulated as follows:

$$\begin{aligned} \min \quad & \sum_{i=1}^{5} |f_i(\mathbf{X},\mathbf{U})| \\ \text{s.t.} \quad & |f_{R_1}(\mathbf{X},\mathbf{U})| \leqslant [1, 1.3], \\ & |f_{R_2}(\mathbf{X},\mathbf{U})| \leqslant [0.9, 1.2], \\ & \mathbf{X} \in \Omega, \end{aligned} \tag{21}$$

where $\mathbf{X}$ is a 6-dimensional decision vector composed by the coordinates of the three locators in x and y directions. $\mathbf{U}$ is a 3-dimensional uncertain vector composed by the spot-welding force, Young's modulus and Poisson's ratio of the sheet metal. f_i represents the displacement of the ith key point in z direction. f_{R_1} and f_{R_2} denote the displacements of the points R_1 and R_2 in z direction, respectively. Ω represents a 500×400 rectangular field of the sheet metal (excluding the two locating holes).

The FEM is used to simulate the deformation of the sheet metal. The shell element is employed to create the FEM mesh. The locators are simplified as point contact with the sheet metal and the six degrees of freedom (DOFs) of the nodes at the locators are all constraint. The displacement DOFs of the nodes which locate in the circumference of the locating holes are constraint in x and y directions.

ξ is specified as 0. Based on Eq. (17), the normalization factors ϕ and ψ are specified as 1.5 and 3.0, respectively. For IP-GA, the search parameters r, s, t, the population size and the probability of crossover are set to 0.6, 0.6, 0.6, 5 and 0.5, respectively. The maximum generations are set to 300. The weighting factor β and penalty factor σ are given 0.5 and 1000, respectively. The constraints are given a same possibility degree level λ. The possibility degrees of these two inequality constraints are denoted by P_1 and P_2, respectively.

Table 5
Optimization results of the locators under different possibility degree levels

λ	P_1	P_2	Penalty function $\tilde{f}$	The optimal positions of the three locators ($z=0$) (mm)
0.0	0.11	0.25	1.31	(155.9, 317.8), (469.4, 346.7), (382.4, 89.40)
0.2	0.45	0.21	2.42	(66.70, 274.5), (184.3, 47.10), (325.5, 346.7)
0.4	0.60	0.58	4.28	(482.1, 232.8), (18.70, 185.9), (214.5, 12.60)
0.6	0.69	0.80	6.15	(18.60, 171.8), (256.9, 22.50), (496.1, 324.7)
0.8	0.88	0.92	8.45	(168.3, 75.80), (492.2, 291.8), (113.5, 317.5)
1.0	1.00	0.96	72.78	(497.5, 123.5), (329.4, 3.140), (214.8, 129.9)

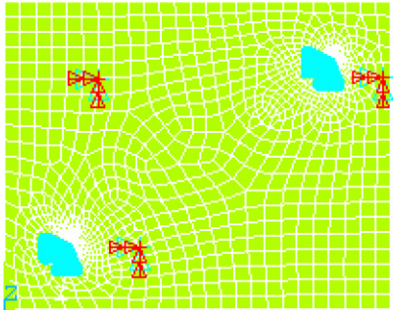

Fig. 8. Optimal arrangement of the locators with $\lambda = 0.8$.

Four GA runs with different random number seeds -1000, -5000, $-10{,}000$ and $-15{,}000$ are performed, and the mean values of these optimization results are listed in Table 5. It can be found that the penalty function $\tilde{f}$ has the largest value 72.78 when $\lambda = 1.0$, and the optimal positions of the three locators are (497.5, 123.5), (329.4, 3.140) and (214.8, 129.9), respectively. With the decreasing of λ, $\tilde{f}$ becomes smaller and smaller. When $\lambda = 0$, $\tilde{f}$ reaches a best value 1.31 and this is actually a non-constraint problem. The corresponding optimal positions of the three locators are (155.9, 317.8), (469.4, 346.7) and (382.4, 89.40), respectively. For $\lambda = 0.2$, 0.4, 0.6 and 0.8, the penalty function $\tilde{f}$ is 2.42, 4.28, 6.15 and 8.45, respectively. Fig. 8 is the optimal arrangement of the locators with $\lambda = 0.8$ in which the arrows represent the locators.

The optimization results also imply that the better design objective and the larger possibility degrees of the constraints cannot be obtained simultaneously. For some practical engineering problems in which the design objective is most cared and the constraints are permitted to be violated at certain extent, a relatively small satisfactory degree level can be selected; for some problems in which the reliability and security are most important, then a relatively large satisfactory degree level should be specified. Additionally, the attitude of the decision maker will also influence the selection of the possibility degree level. An optimistic decision maker usually uses a small satisfactory degree level, while a pessimistic decision maker intends to use a relatively large one.

6. Conclusion

In this paper, an NINP method is suggested to deal with the uncertain optimization problems. A general uncertain optimization model is studied in which the objective function and the constraints are both nonlinear and uncertain, and not only inequality constraints but also equality constraints are investigated. The uncertain single-objective problem is converted to a deterministic two-objective problem, which considers both of the average value and the robustness of the design. A modified possibility degree of interval number based on the probability method is suggested to deal with the inequality constraints, and the equality constraint is solved by transforming it into two inequality constraints. IP-GA with fine global convergence performance is employed as optimization solver.

In the benchmark test, the different generation numbers are used to test the convergence performance of the present method. For each generation number, four GA runs are performed using the different random number seeds and their mean values are regarded as the optimal solutions, and hence the higher confidence about the global solution can be ensured. The computation results indicate that the optimization solutions will reach the stable optimums after certain generation number, and the convergence velocity of IP-GA is very high. Additionally, the application to the locators' optimization in an automobile welding fixture shows the capacity of the present method to deal with the actual engineering problem. From these two numerical examples, it is found that generally the better design objective and the larger possibility degrees of constraints cannot be obtained simultaneously. Thus a trade-off should be made between them according to the practical engineering problem through adjusting the possibility degree level. The present method provides a new approach to deal with a general nonlinear optimization problem with uncertainty. It can substitute stochastic optimization methods to solve some uncertain problems in which the enough information on the uncertainty is unavailable.

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